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GENERALIZED KdV AND QUANTUM INVERSE SCATTERING DESCRIPTION OF CONFORMAL MINIMAL MODELS

D. Fioravanti, F. Ravanini and M. Stanishkov*

*I.N.F.N. Sect. and Dept. of Physics - Univ. di Bologna
Via Irnerio 46, I-40126 BOLOGNA, Italy*

Abstract

We propose an alternative description of 2 dimensional Conformal Field Theory in terms of Quantum Inverse Scattering. It is based on the generalized KdV systems attached to $A_2^{(2)}$, yielding the classical limit of Virasoro as Poisson bracket structure. The corresponding T-system is shown to coincide with the one recently proposed by Kuniba and Suzuki. We classify the primary operators of the minimal models that commute with all the Integrals of Motion, and that are therefore candidates to perturb the model by keeping the conservation laws. For our $A_2^{(2)}$ structure these happen to be $\phi_{1,2}, \phi_{2,1}, \phi_{1,5}$, in contrast to the $A_1^{(1)}$ case, studied by Bazhanov, Lukyanov and Zamolodchikov [1], related to $\phi_{1,3}$.

* Address after June 1, 1995: I.N.R.N.E. - Sofia, Bulgaria

E-mail: fioravanti@bo.infn.it, stanishkov@bo.infn.it, ravanini@bo.infn.it

1. The great success of two-dimensional Conformal Field Theory (CFT) in the last years is mainly due to its large symmetry. It possesses in fact an infinite number of conserved charges closing a kind of generalization of W_∞ algebra. Actually, it was shown [2] that even the Virasoro subalgebra generated by the higher momenta of the stress-energy tensor is often enough for classifying the fields present in the theory and for the computation of their 4-point correlation functions. Perturbation of CFT with some relevant operator leads the system out of the scale invariant fixed (or critical) point. For specific perturbations, an infinite number of commuting conserved charges survives [3], thus leading to an integrable theory. The classification of all the possible integrable perturbations of a given CFT is an important open question. At present, little is known about the integrable field theories themselves. One of the main reasons for this is that, despite the presence of an infinite number of conserved charges, these generate an abelian algebra from which it is difficult to extract information about correlation functions and physical quantities in general. It is well known, however, that the integrable field theories describe effectively a factorized scattering theory (FST) of (massive [4] or massless [5]) particles. The on-shell information, including the asymptotic states created by the so-called Zamolodchikov-Faddeev (ZF) operators $Z_a(\theta)$ and the factorized S-matrices, is available at present for a large class of such FST. Understanding the relation between the two descriptions of the integrable field theory and in particular between the ZF operators and the corresponding operators of the underlying Quantum Field Theory is an important open problem.

On the other hand, CFT itself is an integrable theory – it can be shown in fact that the large symmetry algebra mentioned above has (at least one) infinite abelian subalgebra. A possible strategy, put forward recently by Bazhanov, Lukyanov and Zamolodchikov [1] consists in trying to understand the aforementioned link between the particle description and the field theory one by implementing a Quantum Inverse Scattering Method (QISM) for CFT. Of course, if the interest is CFT alone, this investigation seems quite academic, as there are simpler and more efficient ways to solve it. However, this approach may become very fruitful if one thinks to “prepare” the CFT to be perturbed off-criticality in an integrable direction. Indeed, off-criticality the Virasoro symmetry is lost, and the QISM results in the most hopeful method today available to compute physical quantities of the theory. The proposal of [1] is then first to map the CFT data into a QISM structure *at criticality* and later study how to leave the critical point by suitable “perturbation” of this structure. Only the first part of this project is considered in [1] and in the present paper.

2. The starting point of [1] is the classical limit ($c \rightarrow -\infty$) of CFT given by the KdV system [6, 7], which corresponds to $A_1^{(1)}$ in the Drinfeld-Sokolov [8] classification¹,

¹We recall that in [8] a generalized *modified* KdV (mKdV) hierarchy is attached to each affine Lie Algebra \mathcal{G} . Various Miura transformations relate it to the generalized KdV hierarchies, each one classified by the choice of a node c_m of the Dynkin diagram of \mathcal{G} . Nodes symmetrical under some

and describes the isospectrality condition of the second order differential operator

$$L = \partial_u^2 + U(u) - \lambda^2 \quad (1)$$

The classical limit is realized as $T(u) \rightarrow -\frac{c}{6}U(u)$, $T(u)$ being the stress-energy tensor on a periodic strip ($T(u + 2\pi) = T(u)$) and the Virasoro algebra is turned into the Poisson bracket algebra

$$\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u - v) + \delta'''(u - v) \quad (2)$$

Eq.(1) has a first order matrix representation

$$\mathcal{L} = \partial_u - \phi'(u)h - (e_0 + e_1) \quad (3)$$

where the $A_1^{(1)}$ generators e_0, e_1, h in the fundamental representation and canonical gradation read

$$e_0 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

$\phi(u)$ is related to $U(u)$ by the Miura transformation $U(u) = -\phi'(u)^2 - \phi''(u)$.

Of primary interest is the monodromy matrix $\mathbf{M}_j(\lambda)$ of the operator (3), computed in a generic irreducible representation π_j , $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and especially its trace $\mathbf{T}_j(\lambda)$, which is known to be a generating function for the commuting integrals of motion. The quantization procedure [9] consists essentially in using the quantum deformations $(A_1^{(1)})_{q_{\pm}}$ instead of $A_1^{(1)}$, where

$$q_{\pm} = e^{i\pi\beta_{\pm}^2}, \quad \beta_{\pm} = \sqrt{\frac{1-c}{24}} \pm \sqrt{\frac{25-c}{24}}, \quad \beta_+ = \frac{1}{\beta_-} \quad (5)$$

and a free scalar field

$$\phi(u) = Q + Pu + i \sum_{n \neq 0} \frac{a_n}{n} e^{inu} \quad (6)$$

$$[Q, P] = i\frac{\beta_{\pm}^2}{2}, \quad [a_n, a_m] = \frac{\beta_{\pm}^2}{2} n \delta_{n, -m} \quad (7)$$

The Miura transformation translates, at the quantum level, into the celebrated Feigin-Fuchs construction of the CFT through the screened free boson (6)

$$-\beta_{\pm}^2 T(u) =: \phi'(u)^2 : + (1 - \beta_{\pm}^2) \phi''(u) + \frac{\beta_{\pm}^2}{24} \quad (8)$$

The two possible choices of parametrising with β_{\pm} correspond to the two $(A_1)_{q_{\pm}}$ structures known to be present in $M_{p,p'}$, namely $q_- = e^{i\pi\frac{p}{p'}}$ and $q_+ = e^{i\pi\frac{p'}{p}}$ [10]. In [1] only automorphism of the Dynkin diagram lead, however, to the same hierarchy. The classical Poisson structure of this hierarchy is a classical $w(\tilde{\mathcal{G}})$ -algebra, where $\tilde{\mathcal{G}}$ is the Lie algebra obtained by deleting the c_m node.

the β_- parametrisation was considered. From $c = 1 - 6(\beta_+ + \beta_-)$ one can see that the classical limit $c \rightarrow -\infty$ can be equally well realized by sending $\beta_+ \rightarrow \infty$, $\beta_- \rightarrow 0$ (the choice in [1]), or vice-versa.

The quantum version of the trace of the monodromy matrix $\mathbf{T}_j(\lambda)$ happens to be an entire function of λ^2 , the coefficients in the expansion being given by the “non-local IM”. It exhibits an essential singularity at infinity. The corresponding asymptotic expansion generates the well known “local IM” of CFT. One can compute perturbatively the first few terms in the expansion and check that these \mathbf{T} -operators (as well as their eigenvalues) satisfy a functional relation

$$\mathbf{T}_j(q_{\pm}^{1/2}\lambda)\mathbf{T}_j(q_{\pm}^{-1/2}\lambda) = 1 + \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda) \quad (9)$$

When β_{\pm}^2 is a rational number ($\frac{p}{p'}$ or $\frac{p'}{p}$), q_{\pm} are roots of unity and quantum group truncation takes effect, reducing the system (9) to a finite one. The main conjecture of [1] is that the states in the Hilbert space of the $M_{p,p'}$ minimal model are in one to one correspondence with the solutions of the corresponding truncated T-system having suitable analytic properties (see [1] for details). Thus, the T-system describes completely the chiral Hilbert space in the particle description.

The two objects $\int_0^{2\pi} du : e^{\pm 2\phi} :$ both commute with the local IM [11, 12, 7]. One of them can be chosen to be, in the Feigin-Fuchs construction language, the screening charge. There are two possible choices for screening charges, which amount to choose one of the two parametrisations with β_{\pm} . The β_- choice of [1] identifies the other vertex operator with the $\phi_{1,3}$ one, as a simple calculation of conformal dimensions shows [12]. Perturbations by an operator whose charge commutes with a list of IM, means that, although modified, the whole series of IM continues to be conserved off-criticality. In other words, the structure (9) should be kept by the perturbation, although the analytic properties of the solutions should change drastically. The β_- choice then can be seen as the most efficient QISM description of the particle structure “ready” to describe perturbation by the $\phi_{1,3}$ relevant operator. It is interesting to note that the other choice (β_+) leads instead to the “perturbing” operator $\phi_{3,1}$. This latter is never relevant. However, it is known that this operator is “formally” integrable and has the important role of attracting towards infrared the integrable fluxes $M_{p,p'} \rightarrow M_{2p-p',p}$. The application of this observation to massless integrable theories has yet to be explored. We only mention here that if we apply the truncation of (9) as explained in sect. 5 to this situation, we get a Y-system *formally* identical to the one describing the flux $M_{p+1,p+2} \rightarrow M_{p,p+1}$.

3. In what follows we give an example of the non-uniqueness of this description of CFT as integrable theory. Indeed, consider the generalized KdV equations corresponding to the two vertices c_0 and c_1 of the Dynkin diagram of $A_2^{(2)}$ in the Drinfeld-Sokolov [8] classification

$$\begin{aligned} c_0 & : \quad \partial_t U = \partial_u^5 U + 5U\partial_u^2 U + 5\partial_u U\partial_u^2 U + 5U^2\partial_u U \\ c_1 & : \quad \partial_t U = \partial_u^5 U + 10U\partial_u^3 U + 25\partial_u U\partial_u^2 U + 20U^2\partial_u U \end{aligned} \quad (10)$$

As the usual KdV, both equations (10) are Hamiltonian. Their second Hamiltonian structures are associated with the Hamiltonians

$$H^{(0)} = 3(\partial_u U)^2 - 16U^3 \quad , \quad H^{(1)} = 3(\partial_u U)^2 - U^3 \quad (11)$$

Here and in the following, the superscript in parenthesis $^{(0)}$ and $^{(1)}$ refer to the c_0 and c_1 cases respectively. The crucial observation is that the Poisson bracket algebra of the fields $U(u)$ corresponding to these two second hamiltonian structures coincides again with the classical ($c \rightarrow -\infty$) limit of the Virasoro algebra, eq.(2). The systems (10) describe isospectral deformations of *third* order differential operators

$$L^{(0)} = \partial_u^3 + U\partial_u + \partial_u U - \lambda^3 \quad , \quad L^{(1)} = \partial_u^3 + U\partial_u - \lambda^3 \quad (12)$$

Eqs. (10) can be obtained directly by *reduction* of the Boussinesq equation, which describes the classical limit of CFT having extended W_3 -algebra symmetry [7]. There are two consistent reductions of Boussinesq equation: $W = \partial_u U$ and $W = 0$, leading to the first and second equation of (10) respectively. However this observation is valid only at the classical level, since $(A_2^{(2)})_q$, which is relevant for the quantum case, is an essentially nonlinear deformation of $A_2^{(2)}$, and not just a twist of $(A_2^{(1)})_q$. Being integrable, the equations (10) possess an infinite number of conserved IM $I_s^{(i)}$, $i = 0, 1$ having spin $s = 1, 5 \bmod 6$ [13] One can compute them using the Lax pair representations of (10) and show that the Poisson bracket algebra they close is abelian $\{I_k^{(i)}, I_l^{(i)}\} = 0$, $i = 0, 1$. These IM should obviously be the classical limit of the corresponding quantum conserved charges of CFT, and indeed they happen to coincide with the classical limit of the quantum IM written in [7] for the Boussinesq system, once the reductions $W = \partial_u U$ (for c_0) and $W = 0$ (for c_1) are enforced.

Let us consider the first order matrix realization of (12)

$$\mathcal{L} = \partial_u - \phi'(u)h - (e_0 + e_1) \quad (13)$$

Written in the canonical gradation of $A_2^{(2)}$ (see appendix B of [8]) eq.(13) defines the Lax representation for the generalized *modified* KdV (mKdV) corresponding to algebra $A_2^{(2)}$. Now h, e_0, e_1 are the Cartan-Chevalley generators of $A_2^{(2)}$ level 0 algebra

$$e_0 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (14)$$

By choosing instead to represent the h, e_0, e_1 matrices in the two possible standard gradations (c_0 or c_1), one obtains that the first component of eq.(13) satisfies the first and second of (12) respectively.

The expressions (12) are obtained if one takes h, e_0, e_1 in the fundamental representation. One can however give meaning to (13) for general representations of $A_2^{(2)}$. The

irreducible representations π_s relevant here are labelled by an integer $s \geq 0$. From the solution to the equation $\mathcal{L}\Psi(u) = 0$ the monodromy matrix can be easily written

$$\mathbf{M}_s(\lambda) = \pi_s \left\{ e^{2\pi i k h} \mathcal{P} \exp \lambda \int_0^{2\pi} du (e^{-2\phi(u)} e_0 + e^{\phi(u)} e_1) \right\} \quad (15)$$

(recall that the field ϕ has the quasi-periodicity $\phi(u+2\pi) = \phi(u) + 2\pi k$). Its “improved” form $\mathbf{L}_s(\lambda) = \pi_s(e^{-i\pi k h})\mathbf{M}_s(\lambda)$ satisfies the Poisson bracket algebra

$$\{\mathbf{L}_s(\lambda) \otimes \mathbf{L}_{s'}(\mu)\} = [r_{s,s'}(\lambda\mu^{-1}), \mathbf{L}_s(\lambda) \otimes \mathbf{L}_{s'}(\mu)] \quad (16)$$

where $r_{s,s'}$ is the classical r-matrix associated with $A_2^{(2)}$ [14]. It follows from (16) that the trace of the monodromy matrix $\mathbf{T}_s(\lambda) = \text{Tr} \mathbf{M}_s(\lambda)$ closes an abelian Poisson bracket algebra

$$\{\mathbf{T}_s(\lambda), \mathbf{T}_{s'}(\mu)\} = 0 \quad (17)$$

One can check that this \mathbf{T} -operator in the fundamental representation (\mathbf{T}_1) is indeed the generating function of the infinite number of classical IM of the $A_2^{(2)}$ mKdV.

4. Let us turn now to the quantum case. Following [9, 1] we define the quantum monodromy matrix and the \mathbf{L} -operator as follows

$$\mathbf{L}_s(\lambda) = \pi_s \left\{ e^{i\pi P h} \mathcal{P} \exp \lambda \int du (: e^{-2\phi} : q^h e_0 + : e^{\phi} : q^{-h/2} e_1) \right\} \quad (18)$$

$$\mathbf{M}_s(\lambda) = \pi_s(e^{i\pi P h})\mathbf{L}_s(\lambda) \quad (19)$$

where $\phi(u)$ is a free massless scalar field like (6), and e_0, e_1, h are now Cartan-Chevalley generators of the affine quantum algebra $(A_2^{(2)})_q$ for $q = e^{i\pi\beta^2}$:

$$[e_i, f_j] = \delta_{ij}[h_j] \quad , \quad [h_i, e_j] = a_{ij}e_j \quad , \quad [h_i, f_j] = -a_{ij}f_j \quad , \quad i, j = 0, 1$$

$$h = h_0 = -2h_1 \quad , \quad a_{00} = a_{11} = 2 \quad , \quad a_{01} = -4 \quad , \quad a_{10} = -1 \quad (20)$$

where $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$. We shall comment later on the relation between β and c in this case. Similarly to the classical case we can give meaning to (18) in any irreducible representation of $(A_2^{(2)})_q$.

We briefly describe here these representations, as, up to our knowledge, they are nowhere written explicitly in the mathematical literature. Denote the basic vector of the representation π_s as $|j, m\rangle$, $j = 0, \frac{1}{2}, 1, \dots, \frac{s}{2}$, $m = -j, -j+1, \dots, j$. We define the action of the generators of $(A_2^{(2)})_q$ on this basis by

$$\begin{aligned} h|j, m\rangle &= 2m|j, m\rangle \\ e_0|j, m\rangle &= \sqrt{[j-m][j+m+1]}|j, m+1\rangle \\ f_0|j, m\rangle &= \sqrt{[j+m][j-m+1]}|j, m-1\rangle \\ e_1|j, m\rangle &= \sqrt{e(j)[j-m+1]}|j+\frac{1}{2}, m-\frac{1}{2}\rangle + \sqrt{e(j-\frac{1}{2})[j+m]}|j-\frac{1}{2}, m-\frac{1}{2}\rangle \\ f_1|j, m\rangle &= \sqrt{e(j)[j+m+1]}|j+\frac{1}{2}, m+\frac{1}{2}\rangle + \sqrt{e(j-\frac{1}{2})[j-m]}|j-\frac{1}{2}, m+\frac{1}{2}\rangle \end{aligned} \quad (21)$$

and

$$e(j) = \frac{[j+1][j+\frac{1}{2}]}{[\frac{1}{2}][2j+2][2j+1]} \{[\frac{s}{2}+1] + [\frac{s}{2}+\frac{1}{2}] - [j+1] - [j+\frac{1}{2}]\} \quad (22)$$

is the solution of the recursive equation

$$\frac{[2j]}{[j]}e(j-\frac{1}{2}) - \frac{[2j+2]}{[j+1]}e(j) = 1 \quad , \quad e(\frac{s}{2}) = 0 \quad (23)$$

One can verify by direct calculation that the definition (21) indeed ensures the closing of the $(A_2^{(2)})_q$ algebra provided (23) is satisfied.

Let us now return to the operator (18). It can be shown that $\mathbf{L}_s(\lambda)$ so constructed satisfies the quantum Yang-Baxter equation

$$\mathbf{R}_{ss'}(\lambda\mu^{-1})(\mathbf{L}_s(\lambda) \otimes \mathbf{1})(\mathbf{1} \otimes \mathbf{L}_{s'}(\mu)) = (\mathbf{1} \otimes \mathbf{L}_{s'}(\mu))(\mathbf{L}_s(\lambda) \otimes \mathbf{1})\mathbf{R}_{ss'}(\lambda\mu^{-1}) \quad (24)$$

where now $\mathbf{R}_{ss'}$ is the quantum \mathbf{R} -matrix associated with $(A_2^{(2)})_q$. For instance, for the product of fundamental representations $\pi_1 \otimes \pi_1$ one has a 9×9 matrix given in [14].

The definition (18) is understood in terms of power series expansion in λ

$$\mathbf{L}_s(\lambda) = \pi_s \left\{ e^{i\pi Ph} \sum_{k=0}^{\infty} \lambda^k \int_{2\pi \geq u_1 \geq \dots \geq u_k \geq 0} du_1 \dots du_k K(u_1) \dots K(u_k) \right\} \quad (25)$$

where

$$K(u) =: e^{-2\phi(u)} : q^h e_0 + : e^{\phi(u)} : q^{-h/2} e_1 \quad (26)$$

Similarly to the case considered in [1] an estimate of the singularity properties of the integrands shows that the integrals in (25) should be convergent for $\beta^2 < \frac{1}{2}$ and need regularization for $\beta^2 \geq \frac{1}{2}$. The analytic properties of the eigenvalues of \mathbf{T}_j are strongly influenced by this regularization. However, the general structure of the recurrence relations (32), as well as (9), having its roots in purely algebraic properties of affine quantum algebras, is expected to remain intact for all the $c < 1$ CFT's.

A direct consequence of (24) is that the trace of the quantum monodromy matrix

$$\mathbf{T}_s(\lambda) \equiv \text{Tr} \mathbf{M}_s(\lambda) \quad (27)$$

defines a commuting operator $[\mathbf{T}_s(\lambda), \mathbf{T}_{s'}(\mu)] = 0$ which is the generator of quantum local and non-local IM. In the case of the fundamental representation π_1 , one easily computes $\mathbf{T}_1(\lambda)$ in terms of power series expansion around $\lambda = 0$

$$\mathbf{T}_1(\lambda) = 2 \cos 2\pi P + \sum_{n=1}^{\infty} \lambda^{3n} Q_n \quad (28)$$

where

$$\begin{aligned} Q_n &= q^{3n/2} \int_{2\pi \geq u_1 \geq \dots \geq u_{3n} \geq 0} du_1 \dots du_{3n} \times \\ &\times \left\{ e^{2i\pi P} : e^{-2\phi(u_1)} :: e^{\phi(u_2)} :: e^{\phi(u_3)} : \dots : e^{-2\phi(u_{3n-2})} :: e^{\phi(u_{3n-1})} :: e^{\phi(u_{3n})} : \right. \\ &+ \left. e^{-2i\pi P} : e^{\phi(u_1)} :: e^{\phi(u_2)} :: e^{-2\phi(u_3)} : \dots : e^{\phi(u_{3n-2})} :: e^{\phi(u_{3n-1})} :: e^{-2\phi(u_{3n})} : \right\} \end{aligned} \quad (29)$$

are the non-local IM. As a consequence of (28), $\mathbf{T}_1(\lambda)$ is an entire function of λ^3 . One can show that it also exhibits an essential singularity at infinity. The analysis of the corresponding asymptotic expansion should involve a hard Bethe Ansatz calculation that we do not perform here. Even the “rough estimate” [15] proposed in [1] for the behaviour of the leading term of the asymptotic expansion becomes much more involved in the present case, as the relevant terms in the expansion of $\mathbf{T}_s(\lambda)$ do not enjoy the same nice symmetries as in [1]. Our expectation, of course, is that the coefficients in this expansion should be given by the quantum version of the local IM [7]. We intend to return on this important problem in the near future [16].

In the general case, eq.(27) can be computed using the so-called **R**-fusion procedure [17]. Here we give only the first non-trivial terms in the λ -expansion

$$\mathbf{T}_s(\lambda) = \frac{\sin \frac{s+2}{2}x \sin \frac{s+1}{2}x}{\sin x \sin \frac{x}{2}} + \lambda^3 A_s(x, a) Q_1 + O(\lambda^6) \quad (30)$$

where $x = 2\pi P$, $a = \pi\beta^2$ and

$$\begin{aligned} A_s(x, a) &= \sum_{l=0}^s \frac{1}{8 \sin x \sin a \sin \frac{a}{4}} \left[\frac{\sin(x-a)(l+1)}{\sin(x-a)} - \frac{\sin(x+a)(l+1)}{\sin(x+a)} \right] \\ &\times \frac{\cos \frac{a}{2} \sin \frac{a}{2}(s + \frac{3}{2}) - \cos \frac{a}{2} \sin \frac{a}{2}(l+1)}{\cos \frac{a}{2}(l+1) \cos \frac{a}{2}l} \end{aligned} \quad (31)$$

One can show, using the explicit form (30) that $\mathbf{T}_s(\lambda)$ satisfies (at least to order λ^3) the fundamental relation

$$\mathbf{T}_s(q^{1/6}\lambda)\mathbf{T}_s(q^{-1/6}\lambda) = \mathbf{T}_{s+1}(\lambda)\mathbf{T}_{s-1}(\lambda) + \mathbf{T}_s(q^{1/3\beta^2}\lambda) \quad (32)$$

The very nice result is that this equation coincides with the one recently conjectured in a completely different fashion in [18], and already present essentially in [19]. Conversely, it is interesting to note that, assuming (32) as correct and expanding in λ , one gets, for each order in the expansion, new curious identities. Their meaning and possible relations with fermionic representations of characters have still to be clarified.

The possible choices of β in the $(A_2^{(2)})_q$ case are dictated by adapting the classical limit of the A_1 Feigin-Fuchs construction to the two possible choices of Miura transformations, labeled by c_0 and c_1 . Moreover, the classical limit can be realized in two ways, sending $\beta_+ \rightarrow \infty$ or $\beta_- \rightarrow \infty$. This gives 4 possibilities in total. One can see that both operators $\int_0^{2\pi} du : e^{-2\phi} :$ and $\int_0^{2\pi} du : e^{\phi} :$ commute with the IM [7]. Following the same reasoning as in the $A_1^{(1)}$ case, if one chooses $: e^{-2\phi} :$ as screening operator, then $: e^{\phi} :$ is the perturbing field. The two parametrizations with β_{\pm} correspond to the two possible choices for the screening operator and give $: e^{\phi} := \phi_{1,2}$ and $: e^{\phi} := \phi_{2,1}$ respectively. However, one is also free to choose $: e^{\phi} :$ as screening operator. This leads to the identification of $: e^{-2\phi} :$ with $\phi_{1,5}$ for β_- and $\phi_{5,1}$ for β_+ . The $\phi_{5,1}$ operator is never relevant, and the same considerations as for the $\phi_{3,1}$ operator apply here. It would be

very interesting to understand which fluxes are attracted by this integrable direction in the $M_{p,p'}$ models. The $\phi_{1,5}$ operator is relevant only for $p < p'/2$, $p' < 6$ [20], but can play important roles in different contexts, see e.g. [21]. It is a sort of “dual” operator to $\phi_{2,1}$, in the same sense as $\phi_{3,1}$ is “dual” to $\phi_{1,3}$.

5. In general (9) and (32) could be considered as recursive relations for $\mathbf{T}_s(\lambda)$. For q root of 1 however the quantum group truncation operates and (9,32) become closed systems of functional equations. This important fact allows the authors of [1] to do a crucial conjecture: the solutions of (9) (and we feel to extend the same argument to (32) of course) having the suitable asymptotic behavior and analytic properties (see [1] for details) are the whole set of eigenvalues $t_s(\lambda)$ of $\mathbf{T}_s(\lambda)$ in the Hilbert space of the model. Therefore systems (9) and (32) are complete descriptions of the (chiral) Hilbert space of $c < 1$ RCFT’s, i.e. minimal models. In [1] a transformation is given to pass from (9) to a form which makes contact with scattering theory, the so-called (scale-invariant limit of) Y-system appearing in the context of Thermodynamic Bethe Ansatz (TBA) equations. We have to signal that the success of this transformation is however limited to the case of $M_{2,2n+3}$ models examined in [1]. Comparison with the recently published [22] TBA system for $\phi_{1,3}$ perturbations of $M_{p,p'}$ other than $M_{2,2n+3}$ shows that in general the relation between the \mathbf{T} -system and the Y-system of TBA should be governed by a more complicate relation than the one proposed in [1]. This point out to the existence of a non-trivial structure (sort of generalized Yangian?) relating the Virasoro irreducible representations building up the chiral Hilbert space of the $M_{p,p'}$ model and the organization of the same space under subspaces diagonalising the \mathbf{T}_j ’s. Its understanding passes through the decomposition of Virasoro structures into \mathbf{T}_j -diagonal ones (or vice-versa), a problem on which light can be shed by use of identities relating bosonic (Virasoro) and fermionic (\mathbf{T}_j -diagonal) forms of characters [23]. Also, this approach points out that the searched particle description could be related to the spectrum generating *spinon bases* recently introduced in RCFT [24]. Work in the direction of clarifying these issues is in progress [16], in this short letter we do not deal with this problem completely. Nevertheless, we would like to mention two cases where the Y-system can be *formally* obtained from the T-system with a simple transformation. The meaning of these and more complicated transformations that we intend to report elsewhere still have to be clarified.

- from the system (9), choosing $Y_j(\theta) = (t_{j+\frac{1}{2}}(\lambda)t_{j-\frac{1}{2}}(\lambda))^{-1}$ and $\lambda = e^{\frac{\xi}{\xi+1}\theta}$ for $j = \frac{1}{2}, 1, \dots, n$ and $Y_0 = Y_{n+\frac{1}{2}} = 0$ one recovers the well known Y-system for *unitary* minimal models perturbed by $\phi_{1,3}$ [25].
- A similar situation works in the case of system (32). Take $Y_s(\theta) = t_s(q^{1/3\beta^2}\lambda)(t_{s-1}(\lambda)t_{s+1}(\lambda))^{-1}$ and $\lambda = e^{\frac{\xi+1}{\xi}\theta}$, $\xi = \frac{p}{p'-p}$. With the β_+ parametrization of the choice : $e^{-2\phi}$: as screening operator this gives the $T_1 \diamond A_k$ systems [26], recently studied in [21] in connection with the $M_{p,2p-1}$ models perturbed by $\phi_{2,1}$.

The same system, with $\lambda = e^{\frac{\xi}{\xi+1}\theta}$ and choice β_- for the $:e^\phi$ screening, gives the other half of the perturbation studied in [21], namely $M_{p,2p+1}$ perturbed by $\phi_{1,5}$. The phase shift appearing on the r.h.s. of (32) luckily happens to be readsorbed in these two cases thanks to the periodicity properties of the solutions of the Y-system.

All the other cases involve much more subtle transformations taking into account the delicate structure of the quantum group truncations. In particular, in the $A_2^{(2)}$ case, this latter is even more involved and not yet completely clarified. This is the reason why it is not possible here to find an analog of the “most simple truncation” reported in [1] for $A_1^{(1)}$. Even the simple models $M_{2,2n+3}$ perturbed by $\phi_{1,2}$ happen to have a Y-system that can be obtained from (32) only after a quite involved transformation.

6. To summarize, by exhausting the possible Drinfeld-Sokolov reductions leading to classical Virasoro as Poisson algebra of their second hamiltonian structure, we have found six integrable operators $(\phi_{1,3}, \phi_{3,1}, \phi_{1,2}, \phi_{2,1}, \phi_{1,5}, \phi_{5,1})$, which seem to exhaust the *generically* integrable primary operators (i.e. integrable for *all* minimal models). Of course there are other integrable perturbations for *specific* minimal models. All those known so far seem however to be explained or by some coincidence of operators, or because the model happens to enjoy a larger symmetry than the pure Virasoro one, and the operator is a generically integrable one for this larger W -symmetry.

More generally, this approach could give a simple method to classify all the generically integrable perturbations of coset models of type $\tilde{\mathcal{G}}_k \times \tilde{\mathcal{G}}_1 / \tilde{\mathcal{G}}_{k+1}$ (complications could arise for non-simply-laced $\tilde{\mathcal{G}}$) in a way similar to the observation recently made by Vaysburd in ref. [20]. The connection with that paper could even go further, giving possibly some key of understanding of the duality observed there between the affine Toda system related to specific perturbed CFT and the affine quantum symmetry of the model.

Of course many problems are still unsolved in this new and promising approach initiated in [1]. We mention the already signaled problem to clarify in general the transformation from the T to the Y-system, thus linking this approach with the recent developments of [22]. Another (related) development is the link with fermionic representation of characters [23] that could reveal for example the presence of eventual spectrum generating dynamical symmetries, in the form of *spinon bases* and shed new light on the structure of correlation functions of the fields of the model, in a way suitable for off-critical extensions.

The completeness of such program is related to extending the above constructions out of the conformal point by “perturbing” the integrable structures described here and in [1]. The results of [27] could help to do some progress in this direction. If this “perturbation” could be not too difficult to imagine (although quite hard to implement) for *massive* theories, it creates new problems for *massless* ones. The particle structure of the conformal point in this case is not related to the ultraviolet theory, but to

the infrared one, thus creating the new problem of perturbing the infrared point by an irrelevant (but integrable!) operator, a very delicate problem indeed [28]. The possibility to define QISM structures related to irrelevant integrable operators could be of help in this case. We hope to return to all these intriguing issues elsewhere [16].

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